

Coherent sheaves on generic compact tori

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Abstract

Let T be a compact complex torus, $\dim T > 2$. We show that the category of coherent sheaves on T is independent of the choice of the complex structure, if this complex structure is generic. The proof is independent of math.AG/0205210, where the same result was proven for K3 surfaces and even-dimensional tori.

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1 Introduction

Let M be a compact Kähler manifold. Category $\mathrm{Coh}(M)$ of coherent sheaves on M is an important invariant of M which has deep physical meaning. For manifolds with ample or anti-ample canonical class, M can be reconstructed from the corresponding derived category $D\mathrm{Coh}(M)$ ([BO], [P]). In physics, the objects of $D\mathrm{Coh}(M)$ are interpreted as certain branes on M . Under Mirror Symmetry, this category corresponds to a triangulated category which is constructed by Fukaya from the symplectic geometry of Mirror dual manifold \hat{M} .

Physicists believe that a Mirror dual of n -dimensional compact complex torus is again an n -dimensional compact complex torus. This is one of the first cases where Mirror Symmetry was known explicitly.

In [V1], we have studied the category $\mathrm{Coh}(M)$ for M a K3 surface or even-dimensional compact torus. We proved that $\mathrm{Coh}(M)$ is independent from M , provided that M is generic in its deformation class. The proof of this statement involves hyperkähler geometry.

Given a hyperkähler structure on M , we consider the corresponding twistor space $\mathrm{Tw}(M)$. To every object $F \in \mathrm{Coh}(M)$, we associate a coherent sheaf \hat{F} on $\mathrm{Tw}(M)$, in such a way that the restriction of \hat{F} to M , identified with a fiber of the twistor projection $\pi : \mathrm{Tw}(M) \rightarrow \mathbb{C}P^1$, gives F . Restricting \hat{F} to another fiber close to M , we obtain a coherent sheaf on a manifold which is deformationally equivalent to M . We are free in the choice of hyperkähler structure, and this allows us to obtain any given deformation M' of M after several iterations of this procedure. This construction gives a functor from $\mathrm{Coh}(M)$ to $\mathrm{Coh}(M')$. This functor is equivalence if M' is also generic.

This proof is not very satisfactory, for several reasons. The equivalence $\mathrm{Coh}(M) \cong \mathrm{Coh}(M')$ is not canonical, because it involves many intermediate choices (such as the choice of a hyperkähler structure, and intermediate hyperkähler structures leading from M to M' as we indicated). This dependence is very difficult to estimate. Also, several important steps of the proof do not involve hyperkähler geometry at all, giving an impression that hyperkähler geometry is mostly irrelevant to the big picture. Finally, the isomorphism $\mathrm{Coh}(M) \cong \mathrm{Coh}(M')$ in [V1] is proven only for even-dimensional tori, because odd-dimensional tori are not hyperkähler. By avoiding the hyperkähler geometry entirely, we are able to produce an isomorphism $\mathrm{Coh}(M) \cong \mathrm{Coh}(M')$ for generic tori of arbitrary dimension > 2 .

We use the following notion of generic.

Definition 1.1: Let T be a compact complex torus, $\dim T \geq 3$. We say that T is generic, if T has no non-trivial subtori, $H^{1,1}(T) \cap H^2(T, \mathbb{Z}) = 0$, and $H^{2,2}(T) \cap H^4(T, \mathbb{Z}) = 0$. As Proposition 2.1 indicates, the set of non-generic tori is a set of measure zero.

In this paper, we prove the following theorem.

Theorem 1.2: Let T, T' be compact complex tori of dimension $n \geq 3$. Assume that T, T' are generic. Then the corresponding categories of coherent sheaves are equivalent: $\text{Coh}(T) \cong \text{Coh}(T')$.

We prove Theorem 1.2 in Section 8.

Notice that in this paper all tori are assumed to have $\dim T \geq 3$. We have dealt with the case of 2-dimensional tori in [V1].

The proof of Theorem 1.2 is based on a concept of “ $SO(2n)/U(n)$ -twistor space” for a torus. Given a compact torus T , with flat Kähler metric g , consider the space S of all complex structures I such that g is Kähler on (T, I) . Clearly, S is isomorphic to a Kähler symmetric space $SO(2n)/SU(n)$ (so-called isotropic Grassmanian space). The natural twistor fibration

$$\text{Tw}(T) \longrightarrow S$$

is holomorphic (Proposition 4.2), and in many ways analogous to the twistor fibration known from the hyperkähler geometry. This analogy can be used to adapt the proof of equivalence $\text{Coh}(M) \cong \text{Coh}(M')$ from [V1] to the present needs.

This paper is independent from [V1], and can be used as introductory reading on the subject, as the proofs are significantly simplified.

We also prove the following results about generic tori.

Theorem 1.3: Let T be a generic torus¹. Then

- (i) All proper complex subvarieties of T have dimension 0.
- (ii) Any holomorphic vector bundle on T admits a flat connection compatible with the holomorphic structure.
- (iii) For any coherent sheaf F on T , the reflexization

$$F^{**} := \text{Hom}(\text{Hom}(F, \mathcal{O}_T), \mathcal{O}_T)$$

is a vector bundle.

Proof: Theorem 1.3 (i) is Proposition 3.1, Theorem 1.3 (ii) is Corollary 6.6, and Theorem 1.3 (iii) is Proposition 3.3. ■

We shall use the following bit of notation, which is due to P. Deligne. Consider a complex vector space V . We consider V as a real vector space $V_{\mathbb{R}}$ with an action of $\mathbb{C}^* = U(1) \times \mathbb{R}^{>0}$. Consider some tensor power W of $V_{\mathbb{R}}$ (for instance, the space $\Lambda^i(V_{\mathbb{R}})$ of real exterior differential forms), and let $W_{\mathbb{C}}$ be its complexification. The group \mathbb{C}^* acts on W by multiplicativity:

$$\rho_W : \mathbb{C}^* \times W \longrightarrow W. \tag{1.1}$$

¹Please notice that everywhere in this paper we assume that $\dim T \geq 3$.

We say that a vector $v \in W_{\mathbb{C}}$ has Hodge type (p, q) if $\rho(r, v) = r^{p+q}v$, and $\rho(\theta, v) = \theta^{p-q}v$, where $r \in \mathbb{R}^{>0}$ and $\theta \in U(1)$ are standard generators in $U(1) \times \mathbb{R}^{>0} = \mathbb{C}^*$.

This definition is compatible with the standard notion of Hodge decomposition.

2 Generic tori

In this Section, we prove that Definition 1.1 is consistent, that is, the set of non-generic tori has measure zero in the corresponding moduli space.

Let R be the space of group homomorphisms $\mathbb{Z}^{2n} \xrightarrow{\varphi} \mathbb{C}^n$ such that $\mathbb{C}^n / \varphi(\mathbb{Z}^{2n})$ is compact, up to $GL(n, \mathbb{C})$ -action. The space R admits a natural open embedding to $(\mathbb{C}^n)^{2n} / GL(n, \mathbb{C})$,

$$R \longrightarrow (\mathbb{C}^n)^{2n}, \quad \varphi \longrightarrow (\varphi(\alpha_1), \varphi(\alpha_2), \dots, \varphi(\alpha_{2n})), \quad (2.1)$$

where α_i are generators of \mathbb{Z}^{2n} . Clearly, R is identified with the moduli space of pairs $(T, \alpha_1, \alpha_2, \dots, \alpha_{2n})$, where T is a torus, and $\alpha_1, \alpha_2, \dots, \alpha_{2n}$ a basis in $H^1(T, \mathbb{Z})$. The space R is called **the marked moduli space of complex tori**. It is a covering over the usual moduli of complex tori, with monodromy $GL(2n, \mathbb{Z})$.

Proposition 2.1: Let $R_{sp} \subset R$ be the set of all tori which are not generic, in the sense of Definition 1.1. Then R_{sp} is a countable union of closed complex subvarieties of positive codimension in R .

Proof: Given a proper sublattice $L \subset \mathbb{Z}^{2n}$, $\text{rk } L = 2l$, let $Z_L \subset R$ be the set of all homomorphisms $\mathbb{Z}^{2n} \xrightarrow{\varphi} \mathbb{C}^n$ such that $\varphi(L)$ lies inside a complex space of rank l . For any $\alpha \in \Lambda_{\mathbb{Z}}^2(\mathbb{Z}^{2n})$, or in $\Lambda_{\mathbb{Z}}^4(\mathbb{Z}^{2n})$, denote by Z_{α} the set of all homomorphisms $\mathbb{Z}^{2n} \xrightarrow{\varphi} \mathbb{C}^n$ such that the element corresponding to α in the cohomology of the corresponding torus T is of Hodge type $(1, 1)$ or $(2, 2)$. Clearly, $R_{sp} = \bigcup Z_L \bigcup Z_{\alpha}$, where the union is taken over all L and α . Clearly, Z_L, Z_{α} are closed complex subvarieties of R . To prove Proposition 2.1, it remains to show that $\text{codim } Z_L > 0, \text{codim } Z_{\alpha} > 0$, for all L and α . Since R is irreducible, we have $\text{codim } Z_L > 0, \text{codim } Z_{\alpha} > 0$ unless $Z_L, Z_{\alpha} = R$ for some L or α . Given a proper lattice $L \subset \mathbb{Z}^{2n}$, $\text{rk } L = 2l$, it is easy to find a homomorphism $\mathbb{Z}^{2n} \xrightarrow{\varphi} \mathbb{C}^n$ such that $\mathbb{C} \cdot \varphi(\alpha)$ generates a subspace of dimension $\min(2l, n)$. Therefore, $Z_L \neq R$, and Z_L has positive codimension. To prove that Z_{α} has positive codimension, we need to find a torus T with $\alpha \notin H^{p,p}(T)$ for a given α . We use the following (more general) lemma.

Lemma 2.2: For any n , there exists an n -dimensional compact complex torus T which satisfies

$$H^{p,p}(T) \cap H^{2p}(T, \mathbb{Z}) = 0 \quad \text{for all } 0 < p < n. \quad (2.2)$$

Proof: We need to show that for each $v \in H^{2p}(T, \mathbb{R})$, there exists a complex structure I on T for which $v \notin H^{2p}(T_I, \mathbb{Z})$. Suppose that for some v this is not true. Let $\varphi \in R$ be a homomorphism $\mathbb{Z}^{2n} \xrightarrow{\varphi} \mathbb{C}^n = V$, and ρ the $U(1)$ -action on $\Lambda_{\mathbb{R}}^{2p} V$ corresponding to the Hodge decomposition as in (1.1). Consider the group G generated by such $U(1)$ -actions. It is easy to see that $G = SL(V, \mathbb{R})$. Then, for v as above, v is $SL(V, \mathbb{R})$ -invariant. As follows from classical invariants theory, any non-trivial $SL(V, \mathbb{R})$ -invariant form on V is proportional to the volume form. This proves Lemma 2.2. ■

Remark 2.3: As the proof of Lemma 2.2 indicates, the set of tori satisfying (2.2) is also a complement of a countable union of positive-codimension subvarieties. We use the weaker version of generic (Definition 1.1) because it is sufficient for our purposes.

3 Subvarieties and vector bundles on a generic torus

3.1 Subvarieties of generic torus

Proposition 3.1: Let T be a generic compact torus, and $Z \subset T$ a closed complex subvariety. Then $\dim Z = 0$ or $Z = T$.

Proof: Replacing Z with its irreducible component, we may assume that Z is irreducible. Assume that Z has smallest dimension among all proper subvarieties of T . Then the singular set $\text{Sing } Z$ is 0-dimensional. Since T is generic, it has no Hodge cycles of dimension 1 and 2. Therefore $\dim_{\mathbb{C}} Z > 2$.

The sheaf $\Omega^1(T)$ is globally generated; therefore, the same is true for $\Omega^1(Z)$. Pick k holomorphic 1-forms τ_1, \dots, τ_k , such that $\gamma := \tau_1 \wedge \dots \wedge \tau_k$ is a non-zero section of the canonical class of Z . Unless the sheaf $\Omega^1(Z)$ is trivial outside of $\text{Sing } Z$, γ has non-trivial zero divisor. This contradicts our assumption that Z has minimal dimension among all subvarieties. Therefore, τ_1, \dots, τ_k are linearly independent everywhere. These sections trivialize $\Omega^1(Z)$ outside of $\text{Sing } Z$.

Let $K := \ker(i^* : \Omega^1(T) \rightarrow \Omega^1(Z))$, where i^* denotes the pullback map. Since $\Omega^1(Z)$ is trivialized, we have $\dim \Omega^1 Z = k$, and $\dim K = n - k$. Consider $K \subset \Omega^1 T$ as a distribution. Then K is integrable, and Z is its integral variety. Passing to a covering of T , we find that Z is determined as a common zero set of a system of linear equations. Therefore, Z is flat, and hence it is a subtorus. This is impossible, because T is generic. ■

3.2 Stable bundles on generic tori

We follow [V1] (see also [Voi]).

Proposition 3.2: Let T be a generic compact Kähler torus, and F a stable reflexive¹ sheaf on T . Then F is a line bundle.

Proof: In [BS], Bando and Siu construct a canonical Hermitian Yang-Mills connection ∇ on the non-singular part of any stable reflexive sheaf on a compact Kähler manifold X , with L^2 -integrable curvature Θ . Such a connection (called admissible Yang-Mills) is unique. In [BS] it is also proven that the L^1 -integrable form $\text{Tr}(\Theta \wedge \Theta)$ (considered as a current on the Kähler manifold X) is closed and represents the cohomology class

$$\text{Tr}[\Theta \wedge \Theta] = c_2(F) - \frac{r-1}{r} c_1^2(F) \quad (3.1)$$

Consider the standard Hodge operator $\Lambda : \Lambda^{p,q}(T) \longrightarrow \Lambda^{p-1,q-1}(T)$ on differential forms. Since ∇ is Yang-Mills, we have

$$\Lambda^2(\Theta^2) \geq 0, \quad (3.2)$$

and the equality is reached only if $\Theta = 0$.² Comparing (3.1) and (3.2), and using $c_1(F), c_2(F) = 0$, we obtain that $\Theta = 0$, that is, ∇ is flat. Since F is reflexive, it is non-singular in codimension 2 (see [OSS]). Therefore, ∇ has no local monodromy around singularities of F . This implies that (F, ∇) can be extended to a flat holomorphic bundle on T . Any reflexive sheaf is normal, that is, equal to a pushforward from any open set $T \setminus Z$, with $\text{codim } Z > 1$, (see [OSS]). Therefore, F is smooth.

We proved that F is a smooth holomorphic bundle admitting a flat unitary connection. Denote by χ the corresponding representation of $\pi_1(T)$. Since F is stable, F cannot be split onto a direct sum of subsheaves. Therefore χ is irreducible. Since $\pi_1(T)$ is abelian, its irreducible representation is necessarily 1-dimensional. We proved that F is a line bundle. ■

3.3 Reflexive sheaves on generic tori

We follow [V1].

Proposition 3.3: Let T be a generic compact Kähler torus, and E a reflexive sheaf on T . Then E is a bundle.

Proof: Since T has no non-trivial integer $(1, 1)$ -cycles, the first Chern class of all sheaves on T vanishes. Therefore, all coherent sheaves on T are semistable. Consider the Jordan-Holder filtration

$$0 = E_0 \subset E_1 \subset \dots \subset E_n = E$$

¹A sheaf is called **reflexive** if a natural map $F \longrightarrow \text{Hom}(\text{Hom}(F, \mathcal{O}_T), \mathcal{O}_T)$ is isomorphism.

²This is the celebrated Lübke inequality [Lü], which implies flatness of stable bundles with zero Chern classes and the Bogomolov-Miyaoka-Yau inequality; see [BS].

with all the subfactors E_i/E_{i-1} stable. The reflexizations $(E_i/E_{i-1})^{**}$ are all smooth, by Proposition 3.2. Replacing E_i by its reflexization $E_i^{**} \subset E$, we may assume that all E_i are reflexive. Using induction, we may assume also that E_{n-1} is smooth.

We have an exact sequence

$$0 \longrightarrow E_{n-1} \longrightarrow E \longrightarrow E_n/E_{n-1} \longrightarrow 0$$

with E_{n-1} smooth, E reflexive, and $F = E_n/E_{n-1}$ a stable sheaf having (as we have shown above) a smooth reflexization.

Then E is given by a class $\nu \in \text{Ext}^1(F, E_{n-1})$. Consider the exact sequence

$$0 \longrightarrow F \longrightarrow F^{**} \longrightarrow C \longrightarrow 0$$

where C is a torsion sheaf (cokernel of the reflexization map). This gives a long exact sequence

$$\text{Ext}^1(F^{**}, E_{n-1}) \longrightarrow \text{Ext}^1(F, E_{n-1}) \xrightarrow{\delta} \text{Ext}^2(C, E_{n-1}) \quad (3.3)$$

The kernel of δ in (3.3) corresponds to all extensions $\gamma \in \text{Ext}^1(F, E_{n-1})$ with a reflexization isomorphic to an extension of F^{**} with E_{n-1} . Clearly, such extensions are reflexive only if $C = 0$. To prove that $C = 0$, it suffices to show that $\delta(\nu) = 0$. However, C is a torsion sheaf, and by Proposition 3.1 its support $\text{Supp}(C)$ is a finite set. By Grothendieck's duality ([H], Theorem 6.9), the group $\text{Ext}^2(C, E_{n-1})$ vanishes, for $\text{codim Supp } C > 2$. Therefore, E is smooth. This proves Proposition 3.3. ■

4 $SO(2n)/U(n)$ -twistor space

Let (V, g) be a Euclidean vector space, $\dim_{\mathbb{R}} V = 2n$, and S the set of all Hermitian structures on V compatible with g . Clearly, $S \cong SO(2n)/SU(n)$. The space S is a Kähler symmetric space, which can be seen from the following argument.

Consider the space $V_{\mathbb{C}} := V \otimes \mathbb{C}$, and let S' be the Grassmanian space of all complex subspaces $V_1 \subset V_{\mathbb{C}}$ such that $\dim_{\mathbb{C}} V_1 = n$, and $g_{\mathbb{C}}|_{V_1} = 0$, where $g_{\mathbb{C}}$ is the complexification of g . Since g is positive definite, $V_1 \cap V = 0$. Therefore, the natural projection $V_{\mathbb{C}} \longrightarrow V$, $v \longmapsto \text{Re}(v)$ induces an isomorphism $V_1 \cong V$. The space $V_{\mathbb{C}}$ is equipped with a Hermitian structure as a complexification of Euclidean space. This gives a Hermitian structure on V . It is easy to check that this Hermitian structure is compatible with the Euclidean structure on V . Conversely, any such structure defines a subspace $V^{1,0} \subset V_{\mathbb{C}}$ of vectors of Hodge type $(1,0)$, which is obviously isotropic. We obtain a bijection $S' \cong S$. However, S' is by construction a Kähler symmetric space.

Definition 4.1: We call $S = SO(2n)/U(n)$ the isotropic Grassman manifold.

Given a torus $T := V/\mathbb{Z}^{2n}$, we may identify S with the space of complex structures on T compatible with the metric g . Consider the product $T \times S$, equipped with the complex structure as follows.

Let $(t, s) \in T \times S$ be a point. As we have explained, s defines a complex structure I_s on T . Let $\mathcal{I} : T_t T \times T_s S \longrightarrow T_t T \times T_s S$ act as I_s on $T_t T$ and as I_S on $T_s S$, where I_S is the standard complex structure on S . Clearly, \mathcal{I} is an almost complex structure on $T \times S$.

Proposition 4.2: In the above assumptions, \mathcal{I} is integrable.

Proof: Let B^- be the universal bundle on

$$S = \{V^{1,0} \subset V_{\mathbb{C}} \mid g|_{V^{1,0}} = 0\},$$

with the fiber of B^- at a point $V_1 \subset V_{\mathbb{C}}$ identified with $V^{1,0}$. By construction, $B^- \subset V_{\mathbb{C}} \otimes \mathcal{O}_S$ is holomorphic. We have an exact sequence

$$0 \longrightarrow B^- \longrightarrow V_{\mathbb{C}} \otimes \mathcal{O}_S \xrightarrow{\kappa} B^+ \longrightarrow 0 \quad (4.1)$$

Consider the lattice $L \subset V$, $T = V/L$, and let $\alpha_1, \dots, \alpha_{2n}$ be its generators. Consider the corresponding sections $\alpha_1 \otimes 1, \dots, \alpha_{2n} \otimes 1$ of $V_{\mathbb{C}} \otimes \mathcal{O}_S$. Since $B^-|_s$ is an isotropic subspace of $V_{\mathbb{C}} = V_{\mathbb{C}} \otimes \mathcal{O}_S|_s$, for all $s \in S$, no non-trivial linear combination of $\alpha_i \otimes 1$ lies in B^- . Therefore, the sections $\kappa(\alpha_i \otimes 1) \in B^+$ are linearly independent over \mathbb{R} , and the quotient $B^+/\langle \kappa(\alpha_i \otimes 1) \rangle$ is a compact torus, at every point $s \in S$. The total space $\text{Tw}(T)$ of this quotient is a holomorphic fibration over S , and its total space is naturally identified with $(T \times S, \mathcal{I})$. This proves Proposition 4.2. ■

Definition 4.3: Let (T, g) be a compact complex torus equipped with a flat metric. The complex manifold

$$\text{Tw}(T) := (T \times S, \mathcal{I})$$

is called **the $SO(2n)/U(n)$ -twistor space of (T, g)** , or simply twistor space of T . Clearly, $\text{Tw}(T)$ is equipped with a holomorphic projection $\pi : \text{Tw}(T) \longrightarrow S$, and its fibers are identified with (T, I_s) , where $I_s, s \in S$ are complex structure operators on T compatible with g .

5 Twistor transform for $SO(2n)/U(n)$ -twistor space

5.1 Twistor sections for $\text{Tw}(T)$

Definition 5.1: Let T be a compact torus, and

$$\text{Tw}(T) \xrightarrow{\pi} S$$

its $SO(2n)/U(n)$ -twistor space. A twistor section of $\mathrm{Tw}(T)$ is a holomorphic map $S \xrightarrow{i} \mathrm{Tw}(T)$ such that the composition

$$S \xrightarrow{i} \mathrm{Tw}(T) \xrightarrow{\pi} S$$

is identity. We denote the space of twistor sections by $\mathrm{Sec}(T)$. For any point $t \in T$, the map $I \rightarrow (I, t)$ gives a holomorphic section of π . Such sections are called **horizontal twistor sections**.

Proposition 5.2: In the above assumptions, let $\tau : S \rightarrow \mathrm{Tw}(T)$ be a twistor section, and $I, J \in S$ points in

$$S = \{V^{0,1} \subset V_{\mathbb{C}} \mid g|_{V_1} = 0\},$$

such that the corresponding $V^{0,1}$ -spaces do not intersect:

$$V_I^{0,1} \cap V_J^{0,1} = 0. \quad (5.1)$$

Then, in a neighbourhood U of $[\tau] \in \mathrm{Sec}(T)$, every section τ' is uniquely determined by $\tau'(I)$, $\tau'(J)$. This defines an open embedding $U \hookrightarrow (T, I) \times (T, J)$.

Proof: By construction, $\mathrm{Tw}(T)$ is a quotient of the total space $\mathrm{Tot} B^+$ by a lattice. Since S is simply connected, every section $\tau \in \mathrm{Sec}(T)$ is lifted to a holomorphic section of the projection

$$\mathrm{Tot} B^+ \xrightarrow{\tilde{\pi}} S.$$

Therefore, to prove Proposition 5.2 it suffices to show that a section of B^+ is uniquely determined by its values at I and J . This is implied by the following

Lemma 5.3: Let S be an isotropic Grassmanian (Definition 4.1), B^+ be the universal bundle (see (4.1)), and $I, J \in S$ points satisfying $V_I^{0,1} \cap V_J^{0,1} = 0$. Consider the restriction maps

$$H^0(B^+) \xrightarrow{r_I} B^+|_I, \quad H^0(B^+) \xrightarrow{r_J} B^+|_J.$$

Then

$$H^0(B^+) \xrightarrow{r_I \oplus r_J} B^+|_I \oplus B^+|_J. \quad (5.2)$$

is an isomorphism of vector spaces. In other words, a section of B^+ is uniquely determined by its values at I, J .

Proof: Consider the exact sequence

$$0 \rightarrow B^- \rightarrow V_{\mathbb{C}} \otimes \mathcal{O}_S \xrightarrow{\kappa} B^+ \rightarrow 0$$

(4.1). As Lemma 5.4 (below) implies, $H^0(B^-) = H^1(B^-) = 0$. From the corresponding long exact sequence, we obtain that the induced map

$$H^0(V_{\mathbb{C}} \otimes \mathcal{O}_S) \xrightarrow{\kappa} H^0(B^+) \quad (5.3)$$

is an isomorphism. Therefore, $H^0(B^+)$ is naturally identified with $V_{\mathbb{C}}$, and $\dim H^0(B^+) = 2 \operatorname{rk} B^+$. We obtain that the spaces on the left and the right hand side of (5.2) have the same dimension.

To prove that (5.2) is an isomorphism of vector spaces, it suffices to show that $\ker r_I \cap \ker r_J = 0$. This is equivalent to $V_I^{0,1} \cap V_J^{0,1} = 0$, because under the isomorphism (5.3) $\ker r_I$ corresponds to $V_I^{0,1}$ and $\ker r_J$ corresponds to $V_J^{0,1}$. We reduced Lemma 5.3 and Proposition 5.2 to the following algebro-geometric lemma.

Lemma 5.4: Let S be the isotropic Grassmanian (Definition 4.1) and

$$0 \longrightarrow B^- \longrightarrow V_{\mathbb{C}} \otimes \mathcal{O}_S \xrightarrow{\kappa} B^+ \longrightarrow 0 \quad (5.4)$$

the exact sequence defined above. Then $H^0(B^-) = H^1(B^-) = 0$.

Proof: We use the Kodaira-Nakano vanishing theorem. Since B^- is a holomorphic sub-bundle of a trivial bundle, it is negative: $\Theta_{B^-} \leq 0$, where $\Theta_{B^-} \in \Lambda^{1,1}(S) \otimes \operatorname{End}(B^-)$ is the curvature of B^- ([GH]). The form Θ_{B^-} can be computed explicitly as follows:

$$\Theta_{B^-} = A \wedge A^\perp,$$

where

$$A \in \Lambda^{1,0}(S) \otimes \operatorname{Hom}(B^-, B^+) \quad (5.5)$$

is the second form of the exact sequence (5.4), and

$$A^\perp \in \Lambda^{0,1}(S) \otimes \operatorname{Hom}(B^+, B^-)$$

its Hermitian conjugate (see [GH] for details). The second form A can be computed explicitly, as follows.

The tangent bundle $T(S)$ is naturally identified with $\operatorname{Hom}(B^-, B^+)$, because S is a Grassmanian, and a tangent vector to a Grassmanian of planes $V_0 \subset V_1$ is identified with $\operatorname{Hom}(V_0, V_1/V_0)$. Under the identification

$$\Lambda^{1,0}(S) \cong \operatorname{Hom}(B^+, B^-), \quad (5.6)$$

A becomes an identity operator.

$$\begin{aligned} A &\in \Lambda^{1,0}(S) \otimes \operatorname{Hom}(B^-, B^+) \\ &= \operatorname{Hom}(B^+, B^-) \otimes \operatorname{Hom}(B^-, B^+) \\ &= \operatorname{End}(\operatorname{Hom}(B^-, B^+)). \end{aligned}$$

This implies that $\Theta_{B^-} = A \wedge A^\perp$ is strictly negative. Now, by Kodaira-Nakano theorem, $H^i(B^-) = 0$ for all $i < \dim S$. This proves Lemma 5.4. We finished the proof of Proposition 5.2. ■

Remark 5.5: Consider an anticomplex involution $\iota : S \longrightarrow S, I \longrightarrow -I$. Then $\iota \times Id$ is an anticomplex involution of $\text{Tw}(T)$. This involution maps twistor sections to twistor sections. Abusing notation, we denote the corresponding involution of $\text{Sec}(T)$ by ι .

There is a natural real analytic embedding $T \xrightarrow{\gamma} \text{Sec}(T), t \longrightarrow \{t\} \times S$ mapping t to the corresponding holomorphic section. Clearly, $\gamma(T)$ is a fixed point set of the anticomplex involution $\iota : \text{Sec}(T) \longrightarrow \text{Sec}(T)$. By Proposition 5.2, $\text{Sec}(T)$ is locally in a neighbourhood of $\gamma(T)$ identified with a neighbourhood of diagonal in $(T, I) \times (T, -I)$. We shall think of $\text{Sec}(T)$ as of complexification of its totally real submanifold $T = \gamma(T)$.

5.2 Twistor transform for $\text{Tw}(T)$

Let (B, ∇) be a flat bundle on T , and $(\sigma^* B, \sigma^* \nabla)$ its pullback to $\text{Tw}(T)$. Clearly, $(\sigma^* B, \sigma^* \nabla)$ is flat, and therefore holomorphic. This defines a functor from the category of flat bundles on T to the category of holomorphic bundles on $\text{Tw}(T)$:

$$\text{Fl}(T) \longrightarrow \text{HolBun}(\text{Tw}(T)). \quad (5.7)$$

We call this functor **the direct twistor transform**. It turns out that (5.7) is invertible.

The following theorem is adapted from [KV], where a similar result was proven for a twistor space of a hyperkähler manifold.

Theorem 5.6: Let T be a torus, $\text{Tw}(T) \xrightarrow{\pi} S$ its $SO(2n)/U(n)$ -twistor space, and B a holomorphic vector bundle on $\text{Tw}(T)$. Assume that

$$B \Big|_{S \times \{t\} \subset S \times T = \text{Tw}(T)} \text{ is trivial for all } t \in T, \quad (5.8)$$

that is, B is trivial on all horizontal sections of π . Then B is obtained as a twistor transform of a flat bundle (B_{fl}, ∇) on T . Moreover, (B_{fl}, ∇) is unique, and determines an equivalence

$$\text{HolBun}_0(\text{Tw}(T)) \longrightarrow \text{Fl}(T)$$

of the category $\text{HolBun}_0(\text{Tw}(T))$ of holomorphic bundles on $\text{Tw}(T)$ satisfying (5.8) and the category of flat bundles on T .

We prove Theorem 5.6 in Subsection 5.3.

5.3 Real analytic differential operators and holomorphic vector bundles on twistor spaces

We work in assumptions of Theorem 5.6. Let $B \in \text{HolBun}_0(\text{Tw}(T))$ be a holomorphic vector bundle which is trivial on all horizontal sections of twistor projection. Denote by $U_B \subset \text{Sec}(T)$ the space of all twistor sections $S \xrightarrow{\varphi} \text{Tw}(T)$ for which $B \Big|_{\varphi(S)}$ is trivial.

Lemma 5.7: In the above assumptions, U_B is open in $\text{Sec}(T)$.

Proof: We need to show that a small deformation of a trivial bundle on S is again trivial. It is well known that deformations of vector bundles are classified by $\text{Ext}^1(B, B) = H^1(\text{End}(B))$ ([Kob]). If B is a trivial bundle, $\text{End}(B)$ is also trivial. Therefore, Lemma 5.7 is implied by

$$H^1(\mathcal{O}_S) = 0. \quad (5.9)$$

From (5.6) and ampleness of B^+ , we obtain that TS is ample (this is also clear because S is symmetric). Therefore, the canonical class K of S is negative. By Serre's duality,

$$H^1(\mathcal{O}_S) = H^{\dim S - 1}(K)^*.$$

By Kodaira-Nakano theorem, $H^i(K) = 0$ for all $i < \dim S$, because K is negative. This proves Lemma 5.7. ■

Consider the evaluation map $S \times U_B \xrightarrow{ev} \text{Tw}(T)$, and let ev^*B be the pullback of B . Then ev^*B is trivial on all $S \times \{s\} \subset S \times U_B$, and therefore, the pushforward $B_{\mathbb{C}} = \pi_*(ev^*B)$ is a holomorphic vector bundle on U_B .

One should think of U_B as of complexification $T_{\mathbb{C}}$ of T (Remark 5.5). Then $B_{\mathbb{C}}$ becomes a complexification of a real analytic bundle on T underlying B . The holomorphic structure operator on B can be interpreted as a holomorphic differential operator on $B_{\mathbb{C}}$, as follows.

Fix $I \in S$, and let $T_I \subset TU_B$ be a tangent hyperplane field to a holomorphic foliation \mathcal{L} on $\text{Sec}(T)$ with leaves

$$L_t := \{\varphi : S \longrightarrow \text{Tw}(T) \mid \varphi(I) = t\}$$

parametrized by $t \in T$. We can write $L_t = ev_I^{-1}(t)$. Clearly, $B_{\mathbb{C}} \cong ev_I^*(B|_{(T, I)})$. Therefore, $B_{\mathbb{C}}$ is trivialized along the leaves of \mathcal{L} . This defines a trivial holomorphic connection along the leaves of \mathcal{L} :

$$\nabla_I : B_{\mathbb{C}} \longrightarrow B_{\mathbb{C}} \otimes T_I^*. \quad (5.10)$$

Restricting (5.10) to $\gamma(T) \subset \text{Sec}(T)$ (see Remark 5.5), we obtain that T_I^* becomes $\Lambda^{0,1}(T)$, and (5.10) becomes the holomorphic structure operator

$$\bar{\partial} : B \longrightarrow B \otimes \Lambda^{0,1}(T).$$

Writing this operator as in (5.10) allows us to study the dependence of $\bar{\partial}$ from I in a more explicit way.

Since U_B is an open neighbourhood of its totally real submanifold $\gamma(T)$, it contains a Stein neighbourhood of $\gamma(T)$ as Grauert's theorem implies. Replacing U_B by a smaller neighbourhood if necessary, we may assume that U_B is Stein.

Then $B_{\mathbb{C}}, B_{\mathbb{C}} \otimes T_I^*$ are globally generated, and the map (5.10) can be interpreted as a map of \mathcal{O}_{U_B} -modules:

$$\nabla_I : H^0(B_{\mathbb{C}}) \longrightarrow H^0(B_{\mathbb{C}} \otimes T_I^*).$$

Clearly, (5.10) depends holomorphically from the parameter $I \in S$. Therefore, (5.10) can be interpreted as a map

$$\nabla_I : H^0(B_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathcal{O}_S \longrightarrow H^0(B_{\mathbb{C}}) \otimes_{\mathbb{C}} B^+. \quad (5.11)$$

(we use the identification of $T_I^* = \Lambda^{0,1}(T)$ with B^+ obtained earlier).

Let Γ be a vector space, possibly infinite-dimensional. Consider the exact sequence

$$0 \longrightarrow B^- \otimes \Gamma \longrightarrow V_{\mathbb{C}} \otimes \mathcal{O}_S \otimes \Gamma \xrightarrow{\kappa} B^+ \otimes \Gamma \longrightarrow 0 \quad (5.12)$$

(this is (4.1) tensored by Γ). Since $H^0(B^-) = H^1(B^-) = 0$, the long exact sequence associated with (5.12) gives

$$\mathrm{Hom}(\Gamma_1 \otimes \mathcal{O}_S, \Gamma_2 \otimes B^+) \cong \mathrm{Hom}(\Gamma_1 \otimes \mathcal{O}_S, \Gamma_2 \otimes V_{\mathbb{C}} \otimes \mathcal{O}_S),$$

for any vector spaces Γ_1, Γ_2 . Therefore, the map (5.11) lifts, uniquely, to a holomorphic map

$$\nabla_S : H^0(B_{\mathbb{C}}) \otimes \mathcal{O}_S \longrightarrow H^0(B_{\mathbb{C}}) \otimes V_{\mathbb{C}} \otimes \mathcal{O}_S = H^0(B_{\mathbb{C}} \otimes_{\mathcal{O}_{U_B}} \Omega^1(U_B)) \quad (5.13)$$

It is easy to check that ∇_S is in fact a holomorphic connection on $B_{\mathbb{C}}$. Restricting ∇_S to $\gamma(T) \subset U_B$, we obtain a real analytic connection operator ∇ on B . By construction, the $(0,1)$ -part of ∇ is the holomorphic structure operator on B .

We are going to show that ∇ is flat. Indeed, the $(0,2)$ -part of the curvature Θ of ∇ vanishes for all $I \in S$, because $\bar{\partial} = 0$. If we replace I by $-I$, the $(2,0)$ -forms become $(0,2)$ -forms and vice versa. Therefore, the $(2,0)$ -part of the curvature Θ also vanishes. We obtain that Θ is of type $(1,1)$ with respect to all $I \in S$. In other words, Θ is invariant under the natural $U(1)$ -action induced by the complex structure. It is easy to see that such $U(1)$ generate $SO(2n)$. Therefore, Θ is an $SO(2n)$ -invariant 2-form on $2n$ -dimensional space. Using the standard invariants theory, we infer that $\Theta = 0$. Therefore, ∇ is flat. From the construction it is clear that ∇ is unique.

We obtained a functor from $\mathrm{HolBun}_0(T)$ to the category of flat bundles, which is obviously inverse to the twistor transform. This proves Theorem 5.6.

6 Flat connections on semistable bundles

6.1 Deformation theory and flat bundles

Definition 6.1: Let T be a compact complex torus. Given a holomorphic bundle B on T , we say that B is **polystable** if B is a direct sum of line bundles.

Definition 6.2: Let T be a compact complex torus with flat Kähler metric, and $\text{Tw}(T)$ the corresponding twistor space. Fix a point $I \in S$. Denote by Fl_I the category of flat bundles B on $\text{Tw}(I)$ such that $B|_{(T,I)}$ is polystable, and every irreducible subquotient of B admits a Hermitian metric.

We are interested in the category Fl_I because of the following theorem.

Theorem 6.3: Let T be a compact torus, $\text{Tw}(T) \xrightarrow{\pi} S$ its twistor space, and I, J points which satisfy $V_I^{0,1} \cap V_J^{0,1} = 0$ (5.1). Consider the restriction functor from Fl_I to the category $\text{Bun}(T, J)$ of holomorphic bundles on $(T, J) \subset \text{Tw}(T)$:

$$\text{Fl}_I \xrightarrow{\Phi} \text{Bun}(T, J). \quad (6.1)$$

Then (6.1) is equivalence of categories.

We prove Theorem 6.3 in Subsection 6.2. In the present Section, we prove a weaker form of Theorem 6.3. A similar result is proven in [V1].

Proposition 6.4: In assumptions of Theorem 6.3, let \tilde{B} be a holomorphic bundle on (T, J) . Then $B = \Phi(\tilde{B})$, for some flat bundle $\tilde{B} \in \text{Fl}_I$. Moreover, Φ induces a bijection on the set of isomorphism classes of the objects of corresponding categories.

Proof: We use the following lemma

Lemma 6.5: Let T be a compact complex torus, B_1, \dots, B_n flat holomorphic Hermitian vector bundles and B a holomorphic vector bundle with a filtration

$$0 = E_0 \subset E_1 \subset \dots \subset E_n = B,$$

such that $E_i/E_{i-1} \cong B_i$. Then the holomorphic structure on B can be obtained as follows. Identify B with $B_{gr} := \oplus B_i$ as C^∞ -bundle. Then there is a cohomology class ν

$$\nu \in \bigoplus_{i>j} \text{Ext}^1(B_i, B_j) \subset \text{Ext}^1(B_{gr}, B_{gr}) \quad (6.2)$$

such that the holomorphic structure operator in B is written as

$$\bar{\partial} = \bar{\partial}_{gr} + \nu_0, \quad (6.3)$$

where $\bar{\partial}_{gr}$ is the holomorphic structure operator on B_{gr} , and

$$\nu_0 \in \Lambda^{0,1}(\text{End}(B_{gr}))$$

denotes the harmonic (and hence, flat) representative of ν .

Proof: Write the holomorphic structure operator on B as $\bar{\partial} = \bar{\partial}_{gr} + \tilde{\nu}$, where $\tilde{\nu}$ is a $(0,1)$ -form with values in

$$\oplus_{i>j} \Lambda^{0,1}(T, \text{Hom}(B_i, B_j)).$$

The form $\tilde{\nu}$ satisfies the Maurer-Cartan equation

$$\bar{\partial}^2 = \bar{\partial}_{gr}(\tilde{\nu}) + 2\tilde{\nu} \wedge \tilde{\nu} = 0 \quad (6.4)$$

Every automorphism $g \in \text{End } B_{gr}$ acts on $\tilde{\nu}$ as $\tilde{\nu} \longrightarrow g(\tilde{\nu}) + \bar{\partial}_{gr}(g)$ (this is the well-known gauge action). To produce ν with the properties described in Lemma 6.5, we need to find a correct gauge transform.

Consider the group $(\mathbb{C}^*)^n$ acting on B_{gr} by diagonal automorphisms, in such a way that the i -th component α_i of $(\mathbb{C}^*)^n$ acts trivially on $B_j \subset B_{gr}$ for $i \neq j$, and as a multiplication by α_i on B_i .

We shall write the action of $(\mathbb{C}^*)^n$ on $\Lambda^{0,1}(\text{End}(B_{gr}))$ as follows. Let

$$\begin{aligned} \tilde{\nu} &\in \Lambda^{0,1}(\text{End}(B_{gr})) = \oplus_{i,j} \Lambda^{0,1}(B_i, B_j), \\ \tilde{\nu} &:= \sum_{i,j} \tilde{\nu}_{ij}, \quad \tilde{\nu}_{ij} \in \Lambda^{0,1}(B_i, B_j) \end{aligned}$$

If $\alpha \in (\mathbb{C}^*)^n$, $\alpha = \prod_i \alpha_i$, then

$$\alpha(\tilde{\nu}) = \sum_{i,j} \alpha_i \alpha_j^{-1} \tilde{\nu}_{ij}$$

The group $(\mathbb{C}^*)^n$ acts in this fashion on the solutions of Maurer-Cartan equation, and maps every solution to an equivalent one. If $\alpha_j \gg \alpha_i$ for all $i > j$, then α maps

$$\tilde{\nu} \in \oplus_{i>j} \Lambda^{0,1}(T, \text{Hom}(B_i, B_j))$$

to a form which is arbitrarily small.

Consider the local deformation space $\text{Def}(B_{gr})$ for B_{gr} , constructed in [ST]. The above argument implies that every neighbourhood of the point $[B_{gr}] \in \text{Def}(B_{gr})$ contains a bundle which is isomorphic to B .

Let E be a holomorphic vector bundle over a compact Kähler manifold. The local deformation space $\text{Def}(E)$ can be constructed explicitly in terms of Massey products as follows.

One can define the Massey products as obstructions to constructing a solution of the Maurer-Cartan equation (see e.g. [BT], or [May], [Re] for a more classical approach). Locally, $\text{Def}(E)$ is embedded to the vector space $\text{Ext}^1(E, E)$, and the image of this embedding is a germ of all vectors $\theta \in \text{Ext}^1(E, E)$, such that $\theta \wedge \theta = 0$ and all the higher Massey products of θ with itself vanish.

Fix such a vector $\theta \in \text{Ext}^1(E, E)$. We construct the corresponding vector bundle $E_\theta \in \text{Def}(E)$ using the Hodge theory as follows (see e.g. [V0]).

Let $\theta_0 \in \mathcal{H}^{0,1}(\text{Hom}(E, E))$ be the harmonic representative of θ . Using induction, we define

$$\theta_n := -\frac{1}{2} G_{\bar{\partial}} \sum_{i+j=n-1} \theta_i \wedge \theta_j$$

where $G_{\bar{\partial}}$ is the Green operator inverting the holomorphic structure operator

$$\bar{\partial} : \Lambda^{0,k} \otimes E \longrightarrow \Lambda^{0,k+1} \otimes E.$$

on its image. The Green operator $G_{\bar{\partial}}$ is compact. This can be used to show that for θ sufficiently small, the series $\tilde{\theta} := \sum \theta_i$ converges. The vanishing of Massey products is equivalent to the following condition

$$\bar{\partial}\theta_n = -\frac{1}{2} \sum_{i+j=n-1} \theta_i \wedge \theta_j,$$

which is apparent from the definition given in [BT]. In this case, we have

$$\bar{\partial}\tilde{\theta} = -\frac{1}{2}\tilde{\theta} \wedge \tilde{\theta}$$

and $\tilde{\theta}$ is a solution of the Maurer-Cartan equation (6.4). Therefore, the operator $\bar{\partial}_{\tilde{\theta}} = \bar{\partial} + \tilde{\theta}$ satisfies $\bar{\partial}_{\tilde{\theta}}^2 = 0$, and by Newlander-Nirenberg theorem ([Kob], Proposition 4.17, Chapter 1) this operator defines a holomorphic structure on E . On the deformation space $\text{Def}(E) \subset \text{Ext}^1(E, E)$, the point θ corresponds to a bundle $(E, \bar{\partial}_{\theta})$.

Now we return to holomorphic bundles over a compact torus and the proof of Lemma 6.5. We obtain that B is given by some $\nu \in \text{Ext}^1(B_{gr}, B_{gr})$. Since B and $\oplus B_i$ have compatible filtrations, by functoriality we may assume that

$$\nu \in \oplus_{i>j} \text{Ext}^1(B_i, B_j). \quad (6.5)$$

The higher Massey operations in $\text{Ext}^*(B_{gr}, B_{gr})$ vanish, because the bundle B_{gr} is flat (the same proof works as was used in [DGMS]; see also [GM]). Therefore, B can be reconstructed from ν for any ν such that the cohomology class $\nu \wedge \nu$ vanishes. Pick a harmonic representative ν_0 of ν . Since B_{gr} is flat, ν_0 is parallel. Therefore $\nu_0 \wedge \nu_0$ is also parallel, hence harmonic. We obtain that the cohomology class of $\nu_0 \wedge \nu_0$ vanishes if and only if this form vanishes identically.

Starting from a bundle B with a filtration satisfying the assumptions of Lemma 6.5, we have constructed a cohomology class $\nu \in \oplus_{i>j} \text{Ext}^1(B_i, B_j)$, with $\nu \wedge \nu = 0$. The harmonic representative ν_0 of ν satisfies $\nu_0 \wedge \nu_0 = 0$. Therefore, ν_0 is a solution of Maurer-Cartan equation, and $\bar{\partial}_{gr} + \nu_0$ is equivalent to the holomorphic structure operator of B . This proves Lemma 6.5. ■

Let ∇_{gr} be the standard Hermitian flat connection on B_{gr} . The bundle $(B_{gr}, \nabla_{gr} + \nu_0)$ is also flat, because ν_0 is by construction parallel. We obtained the following corollary.

Corollary 6.6: Let B be a holomorphic vector bundle on a compact generic torus T , $\dim_{\mathbb{C}} T > 2$. Then B admits a flat connection compatible with the holomorphic structure.

■

Return to the proof of Proposition 6.4. We use the notation of Lemma 6.5. Let B be a holomorphic bundle on (T, J) , and B_{gr} the associated graded bundle of Jordan-Holder filtration. Then B_{gr} is polystable, and hence admits a flat Hermitian connection (Proposition 3.2). Consider the bundle \tilde{B}_{gr} on $\text{Tw}(T)$, obtained from the flat bundle B_{gr} via twistor transform, and let $R^1\pi_*(\text{End}(\tilde{B}_{gr}))$ be the corresponding direct image sheaf on S . Clearly, $R^1\pi_*(\text{End}(\tilde{B}_{gr}))$ is isomorphic to a sum of several copies of the vertical tangent bundle $T_{vert} \text{Tw}(T) \cong B^+$. Given a holomorphic section $\tilde{\gamma} \in R^1\pi_*(\text{End}(\tilde{B}_{gr}))$, $\gamma^2 = 0$, we take the fiberwise harmonic (hence, flat) representative $\tilde{\gamma}_0$ of $\tilde{\gamma}$. The bundle $(\tilde{B}_{gr}, \bar{\partial}_{gr} + \tilde{\gamma}_0)$ is holomorphic. Consider the 1-form $\nu_0 \in H^1((T, J), \text{End}(\tilde{B}_{gr}))$ associated with B as in Lemma 6.5.

Let $\tilde{\nu}_0$ be a section of $R^1\pi_*(\text{End}(\tilde{B}_{gr}))$ which vanishes in I and is equal to ν_0 at J . Such a section exists by Lemma 5.3 (we surmise that $R^1\pi_*(\text{End}(\tilde{B}_{gr}))$ is isomorphic to a sum of several copies of B^+). Clearly, $\tilde{B} := (\tilde{B}_{gr}, \bar{\partial}_{gr} + \tilde{\nu}_0)$ is a holomorphic bundle with $\tilde{B}|_{(T, I)} = B_{gr}$ and $\tilde{B}|_{(T, J)} = B$. Therefore, \tilde{B} belongs to Fl_I , and $\Phi(\tilde{B}) = B$. This bundle is uniquely determined by B , because $\tilde{\nu}_0$ is uniquely determined by the conditions $\tilde{\nu}_0|_I = 0$, $\tilde{\nu}_0|_J = \nu$ (Lemma 6.5). This proves Proposition 6.4. ■

6.2 Abelian categories of finite length

Definition 6.7: Let \mathcal{C} be an abelian category. An object $B \in \mathcal{C}$ is called **simple** if it has no proper sub-objects: for all $B' \subset B$, either $B' = B$ or $B' = 0$. An object B is called **semisimple** if B is a direct sum of simple objects.

An abelian category \mathcal{C} is called **of finite length** if every object $B \in \mathcal{C}$ is a finite extension of simple objects. Equivalently, \mathcal{C} is of finite length if any increasing or decreasing chain of sub-objects of B stabilizes, for all $B \in \mathcal{C}$.¹

A **length** of $B \in \mathcal{C}$ is a minimal length of such a chain.

Lemma 6.8: Let $\mathcal{C} \xrightarrow{\gamma} \mathcal{C}'$ be a functor of abelian categories of finite length. Assume that γ induces equivalence on the respective subcategories of semisimple objects. Assume, moreover, that γ induces a bijection on the sets of equivalence classes of the objects of $\mathcal{C}, \mathcal{C}'$. Then γ is equivalence.

Proof: The proof is obtained via the trivial diagram-chasing argument. We use an induction by the length of B . Let $B, B' \in \mathcal{C}$ be objects of length $\leq n$. Consider an exact sequence

$$0 \longrightarrow B_0 \longrightarrow B \longrightarrow B_1 \longrightarrow 0$$

¹One also says \mathcal{C} satisfies the ascending and descending chain condition.

with $l(B_1) = n - 1$ and B_0 semisimple. There is a long exact sequence

$$0 \rightarrow \operatorname{Hom}(B', B_0) \rightarrow \operatorname{Hom}(B', B) \rightarrow \operatorname{Hom}(B', B_1) \rightarrow \operatorname{Ext}^1(B', B_0) \rightarrow \dots$$

Applying γ , we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(B', B_0) & \longrightarrow & \operatorname{Hom}(B', B) & \longrightarrow & \operatorname{Hom}(B', B_1) & \longrightarrow & \operatorname{Ext}^1(B', B_0) \\ & & \gamma \downarrow & & \gamma \downarrow & & \gamma \downarrow & & \gamma \downarrow \\ 0 & \longrightarrow & \operatorname{Hom}(\gamma(B'), \gamma(B_0)) & \longrightarrow & \operatorname{Hom}(\gamma(B'), \gamma(B)) & \longrightarrow & \operatorname{Hom}(\gamma(B'), \gamma(B_1)) & \longrightarrow & \operatorname{Ext}^1(\gamma(B'), \gamma(B_0)) \end{array} \quad (6.6)$$

The maps of Ext^1 -groups are isomorphisms because Ext^1 classify the classes of extensions, and we know that γ induces bijection on isomorphism classes of objects. Using induction by length of B, B' , we may assume that all vertical arrows of (6.6) are isomorphisms, except, possibly, the map

$$\operatorname{Hom}(B', B) \xrightarrow{\gamma} \operatorname{Hom}(\gamma(B'), \gamma(B)).$$

Using the snake lemma, we obtain that this map is also an isomorphism. Therefore, γ is equivalence. ■

We apply Lemma 6.8 to prove Theorem 6.3. Simple objects in $\operatorname{Bun}(T, J)$ are line bundles, as Proposition 3.2 implies. Indeed, simple objects must be stable, and stable bundles are line bundles. Clearly, line bundles on T admit flat Hermitian connections.

As a definition of Fl_I implies, simple objects here are line bundles obtained from flat Hermitian line bundles (T, J) . This implies that

$$\Phi : \operatorname{Fl}_I \longrightarrow \operatorname{Bun}(T, J)$$

is an equivalence on the subcategory of semisimple objects. By Proposition 6.4, Φ induces bijection on equivalence classes of objects. Now, Lemma 6.8 implies that it is an equivalence. This proves Theorem 6.3.

7 Singularities of coherent sheaves on generic torus

7.1 Local geometry on $\operatorname{Tw}(T)$

Let T be a compact torus, $\dim T > 1$. Fix a point $x \in T$, and the corresponding point in twistor space $(I, x) \in \operatorname{Tw}(T)$. We shall study the twistor sections passing through $(I, x) \in \operatorname{Tw}(T)$ in a neighbourhood of the horizontal section $s_x = S \times \{x\}$.

Let $L, L' \in S$ be complex structures satisfying $V_L^{0,1} \cap V_I^{0,1} = V_{L'}^{0,1} \cap V_I^{0,1} = 0$ (see (5.1) for details). Consider a point (t, L') close to (x, L') . By Proposition 5.2, there exists a unique twistor section $s : S \longrightarrow \operatorname{Tw}(T)$ passing through (t, L') and (x, I) and close to $s_x = \{x\} \times S$. This section is unique in a small

neighbourhood of $\{x\} \times S$. We define $\tilde{\Psi}_I(t) := s(L) \in (T, L)$. This map is holomorphic and invertible in a sufficiently small neighbourhood of (x, L') . We think of this map as of isomorphism

$$\Psi_I(L, L') : \mathcal{O}_{x,L} \longrightarrow \mathcal{O}_{x,L'} \quad (7.1)$$

of the corresponding local rings. Clearly, $\Psi_I(L, L')$ holomorphically depends on I, L, L' .

Let now B be a bundle on $\text{Tw}(T)$ obtained from the twistor transform (Theorem 5.6). This means that the restriction of B to any horizontal twistor section s_x is trivial. As we have shown, a small deformation of a trivial bundle on S is again trivial. Therefore, the restriction of B to a twistor section close to s_x is trivial as well.

This allows one to extend the map $\Psi_I(L, L') : \mathcal{O}_{x,L} \longrightarrow \mathcal{O}_{x,L'}$ to the space of germs of holomorphic sections of B . We obtain an isomorphism

$$\Psi_I(L, L', B) : B_{x,L} \longrightarrow B_{x,L'}, \quad (7.2)$$

where $B_{x,L}$ and $B_{x,L'}$ are spaces of germs of $B|_{(T,L)}$ and $B|_{(T,L')}$ in x .

Denote the infinitesimal neighbourhood of $s_x \setminus (I, x)$ in $\text{Tw}(T) \setminus (T, I)$ by

$$\text{Tw}(T)_{x,I} = \text{Spec}(\mathcal{O}_{\text{Tw}(T) \setminus (T,I), s_x}).$$

This space is fibered over $S \setminus I$ with fibers isomorphic to a germ of a smooth complex manifold.

The maps (7.1) produce a canonical trivialization of $\text{Tw}(T)_{x,I}$ over $S \setminus I$. Denote the trivialization map by

$$\Phi : \text{Tw}(T)_{x,I} \longrightarrow \text{Spec } \mathcal{O}_0(\mathbb{C}^n) \times (S \setminus I) \quad (7.3)$$

where $\text{Spec } \mathcal{O}_0(\mathbb{C}^n)$ is the germ of \mathbb{C}^n in 0.

Let $\xi : \text{Tw}(T)_{x,I} \longrightarrow \text{Spec } \mathcal{O}_0(\mathbb{C}^n)$ be the composition of Φ and the projection $\text{Spec } \mathcal{O}_0(\mathbb{C}^n) \times (S \setminus I) \longrightarrow \text{Spec } \mathcal{O}_0(\mathbb{C}^n)$. Using the maps $\Psi_I(L, L', B)$ of (7.2), we obtain a trivialization of $B|_{\text{Tw}(T)_{x_0,I}}$ over $S \setminus I$. More precisely, we obtain a bundle B_S over $\text{Spec } \mathcal{O}_0(\mathbb{C}^n)$, and a natural isomorphism

$$B|_{\text{Tw}(T)_{x,I}} \cong \xi^* B_S. \quad (7.4)$$

7.2 The category \mathcal{C}_I

Definition 7.1: Let T be a compact torus, $\text{Tw}(T)$ its $SO(2n)/U(n)$ -twistor space. The category \mathcal{C}_I is defined as follows.

An object of \mathcal{C}_I is a coherent sheaf F on $\text{Tw}(T)$ satisfying the following conditions.

- (i) The reflexization $B := F^{**}$ belongs to Fl_I .

- (ii) The sheaf F is non-singular outside of $A \times S \subset \mathrm{Tw}(T)$, where $A \subset T$ is a finite set.
- (iii) Let $Z := \{A\} \times I \subset (T, I) \subset \mathrm{Tw}(T)$ be the finite subset of $(T, I) \subset \mathrm{Tw}(T)$ corresponding to A , and $j : \mathrm{Tw}(T) \setminus Z \hookrightarrow \mathrm{Tw}(T)$ the natural embedding. Then the canonical homomorphism $F \longrightarrow j_* j^* F$ is an isomorphism.
- (iv) Let $x \in A$ be a point. Consider the trivialization

$$B \Big|_{\mathrm{Tw}(T)_{x,I}} \xrightarrow{\Psi} \xi^* B_S \quad (7.5)$$

constructed in (7.4), where $B = F^{**}$ is the reflexization of F . Then there is a sheaf F_S on $\mathrm{Spec} \mathcal{O}_0(\mathbb{C}^n)$, equipped with an isomorphism $F_S^{**} \cong B_S$, and a trivialization

$$F \Big|_{\mathrm{Tw}(T)_{x,I}} \xrightarrow{\Psi} \xi^* F_S$$

which is compatible with (7.5).

The morphisms of \mathcal{C}_I are morphisms of coherent sheaves.

Theorem 7.2: Let T be a compact torus, $\mathrm{Tw}(T) \xrightarrow{\pi} S$ its twistor space, and $I, L \in S$ complex structures which satisfy $V_L^{0,1} \cap V_I^{0,1} = 0$. Consider the category \mathcal{C}_I constructed above, and let $\mathcal{C}_I \xrightarrow{\Phi} \mathrm{Coh}(T, L)$ be the restriction map. Then Φ is equivalence of categories.

Proof: We construct the inverse functor $\Upsilon : \mathcal{C}_I \longrightarrow \mathrm{Coh}(T, L)$ as follows. Take $F_L \in \mathrm{Coh}(T, L)$. Let B_L be its reflexization, which is smooth by Proposition 3.3, and $B \in \mathcal{C}_I$ the corresponding bundle over the twistor space, which is defined in a canonical way by Theorem 6.3.

Let $x_0 \in T$ be a singular point of F_L . Consider the infinitesimal neighbourhood of

$$\{x_0\} \times (S \setminus I) \subset \mathrm{Tw}(T) \setminus (T, I)$$

in $\mathrm{Tw}(T) \setminus (T, I)$, denoted, as in Subsection 7.1, by $\mathrm{Tw}(T)_{x_0, I}$.

Then B is trivialized over $\mathrm{Tw}(T)_{x_0, I}$, and we may write $B \Big|_{\mathrm{Tw}(T)_{x_0, I}} = \xi^*(B_{L, x_0})$, where B_{L, x_0} is the germ of E_L in x_0 . Let $F \Big|_{\mathrm{Tw}(T)_{x_0, I}} := \xi^*(F_{L, x_0})$ be the sheaf corresponding to F_L . Since F_L has isolated singularities, the sheaves $F \Big|_{\mathrm{Tw}(T)_{x_0, I}}$ and $B \Big|_{\mathrm{Tw}(T)_{x_0, I}}$ are canonically isomorphic outside of $\{x_0\} \times (S \setminus I)$.

Gluing together $F \Big|_{\mathrm{Tw}(T)_{x_0, I}}$ and B , we obtain a sheaf F_0 on $\mathrm{Tw}(T) \setminus (T, I)$. Let A be the singular set of F_L . Then B is equal to F_0 outside of $\{A\} \times S \subset \mathrm{Tw}(T)$, hence F_0 can be extended smoothly from $\mathrm{Tw}(T) \setminus (T, I)$ to a sheaf F_1 on $\mathrm{Tw}(T) \setminus Z$, where $Z = \{A\} \times \{I\}$ is the finite set defined in Definition 7.1. Consider the sheaf $F := j_*(F_1)$, where $j : \mathrm{Tw}(T) \setminus Z \hookrightarrow \mathrm{Tw}(T)$ is the natural embedding. The correspondence $F_L \longrightarrow F$ is clearly functorial.

The sheaf F by construction belongs to \mathcal{C}_I , and satisfies $\Phi(F) = F_L$. Therefore, $F_L \longrightarrow F$ gives an inverse functor of $\mathcal{C}_I \xrightarrow{\Phi} \text{Coh}(T, L)$. We proved Theorem 7.2. ■

The following corollary immediately follows from Theorem 7.2.

Corollary 7.3: Let T be a generic compact torus, equipped with flat Hermitian metric g , and S the set of complex structures compatible with g . Then, for any generic complex structures $I, J \in S$, the categories of coherent sheaves $\text{Coh}(T, I)$ and $\text{Coh}(T, J)$ are isomorphic. ■

8 Moduli of compact tori

Let $T = \mathbb{C}^n / \mathbb{Z}^{2n}$ be a compact torus with a flat Riemannian metric g . We have shown that for all generic complex structures I, J compatible with g , the category $\text{Coh}(T, I)$ is equivalent to $\text{Coh}(T, J)$. To prove that $\text{Coh}(T, I)$ is equivalent to $\text{Coh}(T, J)$ for arbitrary generic complex structures, it suffices to prove the following proposition.

Proposition 8.1: Let T be a compact torus, $T = V / \mathbb{Z}^{2n}$, $V \cong \mathbb{C}^n$, and R its marked moduli space (Section 2). Given $I \in R$, pick a flat Riemannian metric compatible with I , and let $R_g \subset R$ be the set of all complex structures $J \in R$ compatible with g . Take a generic complex structure $I_1 \in R_g$, and repeat the same procedure, obtaining I_2 . Then, after at most 6 iterations, we can obtain any generic complex structure on T from I .

Proof: We use the following Claim. Please note that the flat metrics on T are in one-to-one correspondence with the Euclidean metrics on V .

Claim 8.2: In assumptions of Proposition 8.1, let \mathcal{G} be the set of all Euclidean metrics g_1 on V such that g_1 is compatible with some complex structure $I \in R_g$. Then $g_1 \in \mathcal{G}$ if and only if all eigenvalues of the symmetric matrix $g_1 g^{-1}$ occur in pairs. In other words, $g_1 \in \mathcal{G}$ if and only if there is an orthonormal basis in (V, g) such that g_1 is written in this basis as

$$\begin{pmatrix} \alpha_1 & & & & & \\ & \alpha_1 & & & & \\ & & \alpha_2 & & & \\ & & & \alpha_2 & & \\ & & & & \ddots & \\ & & & & & \alpha_n \\ & & & & & & \alpha_n \end{pmatrix} \quad (8.1)$$

Proof: “If” part. Let $I_1 \in R_g$ be a complex structure such that g_1 is compatible

with I_1 . Then g, g_1 are Hermitian forms on the complex vector space (V, I_1) . A standard linear-algebraic argument implies that the corresponding Hermitian metrics have a common orthogonal basis $z_1, z_2, \dots, z_n \in (V, I_1)$. After rescaling, we may assume that z_1, z_2, \dots, z_n is orthonormal with respect to g . Consider the corresponding basis $z_1, I_1(z_1), z_1, I_1(z_1), \dots$ in V over \mathbb{R} . In this basis, g is written as identity matrix, and g_1 as (8.1). This proves the “if” part of Claim 8.2.

Conversely, assume that $g^{-1}g_1$ has eigenvalues occuring in pairs. Finding a common orthogonal basis, which is orthonormal in g , we find that g_1 is written in this basis as (8.1). Let I_1 act in this basis as

$$\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix} \quad (8.2)$$

Then I_1 is a complex structure compatible with g and g_1 . Therefore, $g_1 \in \mathcal{G}$. This proves Claim 8.2. ■

The following elementary claim is proven in the same way as one proves Proposition 2.1.

Claim 8.3: In assumptions of Claim 8.2, let R_g° be the set of all generic complex structures $I_1 \in R_g$. Then $R_g \setminus R_g^\circ$ has measure zero.

■

Using Claim 8.3, Claim 8.2 and Fubini theorem, we obtain

Corollary 8.4: In the above assumptions, let \mathcal{G}° be the set of all Euclidean metrics $g_1 \in \mathcal{G}$ such that g_1 is compatible with $I_1 \in R_g^\circ$. Then $\mathcal{G} \setminus \mathcal{G}^\circ$ has measure zero.

■

Now we can prove the following lemma.

Lemma 8.5: In the assumptions of Proposition 8.1, let $R_3(I)$ be the set of all $I_3 \in R$ which can be obtained after three iterations of the procedure defined in Proposition 8.1. Then $R \setminus R_3(I)$ has measure zero.

Proof: Let $\mathcal{F} \in \text{End } V$ be the set of all symmetric operators on (V, g) which can be written as UAU^{-1} , where U is orthogonal, and A takes form (8.1). From Claim 8.2 it is clear that $\mathcal{G} = \mathcal{F} \cdot g$. The second iteration of this procedure gives the following trivial claim.

Claim 8.6: In the above notation, let \mathcal{G}_1 be the set of all Euclidean metrics on V which are compatible with some $I_2 \in \mathbb{R}$, with I_2 being compatible with some $g_1 \in \mathcal{G}$. Then $\mathcal{G}_1 = \mathcal{F} \cdot \mathcal{F} \cdot g$.

■

Clearly, every Euclidean metric can be obtained this way; even if A_1 and A_2 commute, $U_1 A_1 A_2 U_1^{-1}$ runs through the set of all symmetric matrices. Indeed, any diagonal matrix can be obtained as a product of two diagonal matrices with eigenvalues occurring in pairs.

Let \mathcal{G}_1° be the set of all metrics compatible with a generic $I_2 \in R$, after a second iteration of the procedure defined in Proposition 8.1. Then $\mathcal{G}_1 \setminus \mathcal{G}_1^\circ$ has measure 0, as one can easily surmise from Corollary 8.4. Then, \mathcal{G}_1 is the set of all Euclidean metrics except, possibly, a measure zero set, and \mathcal{G}_1 is compatible with all $I \in R$ except, possibly, a measure zero set. This proves Lemma 8.5. ■

Return to the proof of Proposition 8.1. Let $I, J \in R$ be generic complex structures, and $R_3(I), R_3(J) \subset R$ the sets associated with I, J as in Lemma 8.5. The sets $R \setminus R_3(I), R \setminus R_3(J)$ have measure zero, as Lemma 8.5 implies. Therefore, $R_3(I)$ and $R_3(J)$ have non-empty intersection. Take a chain

$$I \rightarrow I_1 \rightarrow I_2 \rightarrow I_3 = J_3 \rightarrow J_2 \rightarrow J_1 \rightarrow J$$

of generic complex structures with each successive pair having a Euclidean metric compatible with both. We obtain that J can be obtained from I after 6 iterations of the procedure defined in Proposition 8.1. Proposition 8.1 is proven. ■

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